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THE PENDULUM-LIKE MOTIONS OF A RIGID BODY IN THE GORYACHEV–CHAPLYGIN CASE†

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The motion of a heavy rigid body with a single fixed point in a uniform gravity field is considered. The geometry of the mass of the body and the initial conditions of its motion correspond to the case of Goryachev–Chaplygin integrability [1, 2]. In this case periodic pendulum-like motions exist, corresponding to oscillations or rotations of the body around an axis of dynamic symmetry, occupying a fixed horizontal position. The problem of the orbital stability of such motions is solved. An explicit solution of the linearized equations of the perturbed motion is obtained and it is shown that, in the linear approximation, the oscillations and rotations of the body are orbitally stable, while the non-linear problem of stability is always a resonance problem: for any amplitude of the oscillations (or any angular velocity of rotation) of the body in unperturbed motion its perturbed motion is such that fourth-order resonance occurs (two non-unity multipliers are pure imaginary and equal to $\pm i$). It is shown that, in the non-linear formulation of the problem, the pendulum-like oscillations of the body are always orbitally unstable, while the rotations are stable.

1. INTRODUCTION

Suppose a rigid body has a single fixed point O and moves in a uniform gravity field. The weight of the body is mg and the distance from the fixed point to the centre of gravity is equal to l. Suppose $OXYZ$ is a fixed system of coordinates, the OZ axis of which is directed vertically upwards. Another system of coordinates $Oxyz$ is rigidly connected with the moving body, its axes Ox, Oy and Oz are directed along the principal axes of inertia of the body for the point O, and the corresponding principal moments of inertia are equal to A, B and C. We will denote by x^*, y^* and z^* the coordinates of the centre of gravity in the system $Oxyz$. We will assume that the geometry of the mass of the body corresponds to the Goryachev–Chaplygin case [1–6]. Then, assuming that $A = B = 4C$ and $z^* = 0$, we can put $x^* = l$ and $y^* = 0$ without loss of generality.

We will specify the orientation of the body using the Euler angles, which are introduced in the usual way. The equations of motion have the form [7]

$$4\frac{dp}{dt} - 3qr = 0, \quad 4\frac{dq}{dt} + 3rp = \mu^2\gamma_3, \quad \frac{dr}{dt} = -\mu^2\gamma_2$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3, \quad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1, \quad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2 \tag{1.1}$$

$$p = \frac{d\psi}{dt}\gamma_1 + \frac{d\theta}{dt}\cos\varphi, \quad q = \frac{d\psi}{dt}\gamma_2 - \frac{d\theta}{dt}\sin\varphi, \quad r = \frac{d\psi}{dt}\gamma_3 + \frac{d\varphi}{dt}$$

$$\gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta$$

We have introduced the notation $\mu^2 = mgl/C$.

In the case of Goryachev–Chaplygin integrability, there is a limit on the initial conditions of motion. They must be such that the constant area integral must be equal to zero, i.e. so that the equality

$$4(p\gamma_1 + q\gamma_2) + r\gamma_3 = 0 \tag{1.2}$$

is satisfied.

When condition (1.2) is satisfied the equations of motion (1.1), in addition to the energy integral and the integral $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, also have the additional integral

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$$r(p^2+q^2)-\mu^2 p\gamma_3 = \mu^3 c \quad (c = \text{const}) \tag{1.3}$$

The presence of the additional integral (1.3) enables us to reduce the integration of the equations of motion to quadratures. Numerous investigations (see the monographs [3–6] and the bibliographies they contain) are devoted to the analytical properties of the solutions of Eqs (1.1) and a qualitative analysis of the motion of a body in the Goryachev–Chaplygin case.

When condition (1.2) is satisfied, Eqs (1.1) have solutions corresponding to plane pendulum-like motions of the body, for which

$$\psi = \text{const}, \quad \theta = \pi/2, \quad p = q = 0, \quad r = d\varphi/dt, \quad \gamma_1 = \sin\varphi, \quad \gamma_2 = \cos\varphi, \quad \gamma_3 = 0$$

For these solutions the Oz axis of dynamic symmetry of the body is fixed and occupies a horizontal position, while the motion of the body around this axis is described by the differential equation of a physical pendulum $d^2\varphi/dt^2 + \mu^2\cos\varphi = 0$. Eliminating from consideration motions that are asymptotic to the unstable equilibrium position of a pendulum (for which $\varphi = \pi/2$), we will investigate the orbital stability of the oscillations of arbitrary amplitude in the neighbourhood of the stable equilibrium position ($\varphi = 3\pi/2$) or rotations with an arbitrary angular velocity.

2. HAMILTON'S FUNCTION

If the geometry of the mass of the body corresponds to the Goryachev–Chaplygin case, Lagrange's function is given by the equalities

$$L = T - \Pi, \quad T = \frac{1}{2}C(4p^2+4q^2+r^2), \quad \Pi = mgl\gamma_1$$

Hamilton's function is the sum $T + \pi$, in which the projections p, q and r of the angular velocity of the body are expressed in terms of the generalized momenta $p_\psi, p_\theta, p_\varphi$, defined by the relations

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = 4C(p\gamma_1+q\gamma_2)+Cr\gamma_3, \quad p_\theta = \frac{\partial L}{\partial\dot\theta} = 4C(p\cos\varphi - q\sin\varphi), \quad p_\varphi = \frac{\partial L}{\partial\dot\varphi} = Cr$$

Taking into account the fact that, when the Goryachev–Chaplygin condition (1.2) is satisfied, the quantity p_ψ is equal to zero, we have

$$p = (p_\theta\cos\varphi - p_\varphi \operatorname{ctg}\theta\sin\varphi)/(4C), \quad q = -(p_\theta\sin\varphi + p_\varphi\operatorname{ctg}\theta\cos\varphi)/(4C), \quad r = p_\varphi/C$$

Then introducing the dimensionless variables q_1, q_2, p_1 and p_2 using the canonical transformation (with valency $(C\mu)^{-1}$)

$$\varphi = \frac{3\pi}{2}+q_1, \quad \theta = \frac{\pi}{2}+q_2, \quad p_\varphi = C\mu p_1, \quad p_\theta = C\mu p_2$$

and also the dimensionless time $\tau = \mu t$, we obtain the following expression for Hamilton's function of the Goryachev–Chaplygin problem

$$H = \frac{1}{8}[4+\operatorname{tg}^2 q_2]p_1^2 + \frac{1}{8}p_2^2 - \cos q_1\cos q_2 \tag{2.1}$$

The additional integral (1.3) of the Goryachev–Chaplygin problem in the variables q_i, p_i $(i = 1, 2)$ can be written in the form

$$p_1(\operatorname{tg}^2 q_2 p_1^2 + p_2^2) - 4\sin q_2(\cos q_1 \operatorname{tg} q_2 p_1 - \sin q_1 p_2) = c \tag{2.2}$$

3. FORMULATION OF THE PROBLEM OF THE ORBITAL STABILITY OF PLANE PERIODIC MOTIONS

Solutions for which $q_2 = p_2 = 0$ and q_1, p_1 are described by canonical equations with Hamiltonian $H^{(0)} = {}^1\!/_2 p_1^2 - \cos q_1$ correspond to plane oscillations and rotations of the body around an axis of dynamic symmetry. These equations have the integral $H^{(0)} = h = \text{const}$. When $-1 < h < 1$ the body performs oscillations in the neighbourhood of a stable equilibrium position, for which the centre of gravity of the body lies on the vertical OZ below the fixed point O. When $h > 1$ plane rotations of the body around the Oz axis occur.

Henceforth it will be more convenient to write the unperturbed motion in action-angle variables I, w, as was done previously in [8] when investigating the plane motions of a Kovalevskaya top. In the case of oscillations we put $k_1 = \sin(\beta/2)$, where β is the amplitude of the oscillations ($0 < \beta < \pi$). Then

$$q_1 = 2\arcsin[k_1 \mathrm{sn}(u, k_1)], \quad p_1 = 2k_1 \mathrm{cn}(u, k_1), \quad u = \frac{2K(k_1)w}{\pi} \tag{3.1}$$

where

$$w = \omega_1 \tau + w(0), \quad \omega_1 = \frac{\pi}{2K(k_1)} \tag{3.2}$$

while $k_1 = k_1(I)$ is the inverse of the function

$$I = \frac{8}{\pi}[E(k_1) - (1 - k_1^2)K(k_1)] \tag{3.3}$$

In the case of rotations, we put $k_2^2 = 2(1 + h)^{-1}$. Then

$$q_1 = 2\mathrm{am}(u, k_2), \quad p_1 = \frac{2}{k_2}\mathrm{dn}(u, k_2), \quad u = \frac{K(k_2)w}{\pi} \tag{3.4}$$

where

$$w = \omega_2 \tau + w(0), \quad \omega_2 = \frac{\pi}{k_2 K(k_2)} \tag{3.5}$$

while $k_2 = k_2(I)$ is the inverse of the function

$$I = \frac{4E(k_2)}{\pi k_2} \tag{3.6}$$

We have used the generally accepted notation for elliptic functions and integrals [9].

In unperturbed motion we have $q_2 = p_2 = 0$ and $I = I_0 = \text{const}$, while the variables q_1 and p_1 for a specified value of I_0 are defined by (3.1)–(3.3) in the case of oscillations, and by (3.4)–(3.6) in the case of rotations.

We will put $r_1 = I - I_0$. The problem of the orbital stability of plane oscillations and rotations of a body is equivalent to the problem of their stability with respect to the variables q_2, p_2 and r_1.

4. THE STABILITY OF THE OSCILLATIONS

We will take the quantity $\omega_1\tau$ as the new independent variable. Hamilton's function of the perturbed motion can be represented in the form of the series

$$H = r_1 + h_2 + H_4 + \dots \tag{4.1}$$

The functions h_2 and H_4 are defined by the equalities

$$h_2 = \frac{K(k_1)}{4\pi}(p_2^2 + 4\mu_{02}q_2^2), \quad H_4 = \mu_{20}r_1^2 + \mu_{12}r_1q_2^2 + \mu_{04}q_2^4$$

$$\mu_{02} = 3\mathrm{dn}^2 u + k_1^2 - 2, \quad \mu_{20} = -\frac{\pi[E(k_1) - (1 - k_1^2)K(k_1)]}{16k_1^2(1 - k_1^2)K^2(k_1)}$$

$$\mu_{12} = \frac{1}{4(1 - k_1^2)}[1 - k_1^2 + 3\mathrm{sn}u\mathrm{dn}u(\mathrm{cn}u\mathrm{zn}u - \mathrm{sn}u\mathrm{dn}u)] \tag{4.2}$$

$$\mu_{04} = \frac{K(k_1)}{12\pi}(6k_1^2\mathrm{cn}^2 u + 2k_1^2 - 1)$$

The quantity u is given by the last formula of (3.1), and k_1 corresponds to the unperturbed motion. The function (4.1) is π-periodic in w. The dots in (4.1) denote as set of terms higher than the fifth power in q_2, p_2 and $|r_1|^{1/2}$.

Integral (2.2) for the equations of perturbed motion can also be represented in the form of a series in powers of the quantities q_2, p_2 and r_1

$$g_2 + g_4 + \ldots = c \tag{4.3}$$

where g_n is a form of power n in q_2, p_2 and $|r_1|^{1/2}$ with coefficients that are periodic in w. In this case g_2 is a quadratic form in q_2 and p_2, having the form

$$g_2 = 8k_1\mathrm{cn}u(2k_1^2 - 1 - k_1^2\mathrm{cn}^2u)q_2^2 + 8k_1\mathrm{sn}u\mathrm{dn}uq_2p_2 + 2k_1\mathrm{cn}up_2^2 \tag{4.4}$$

Stability in the first (linear) approximation. In the linearized equations of perturbed motion r_1 = const, while the change of variables q_2 and p_2, if we take the quantity w as the independent variable, is described by the equations

$$\frac{dq_2}{dw} = \frac{\partial h_2}{\partial p_2}, \quad \frac{dp_2}{dw} = -\frac{\partial h_2}{\partial q_2} \tag{4.5}$$

The function h_2 is defined by the first equation of (4.2).

Suppose $\mathbf{X}(w)$ is the matrix of the fundamental solutions of system (4.5), normalized by the condition $\mathbf{X}(0) = \mathbf{E}$, where $\mathbf{E}$ is the second-order identity matrix. The elements $x_{ij}(w)$ of the matrix $\mathbf{X}$ satisfy the equations

$$\frac{dx_{1j}}{dw} = \frac{K(k_1)}{2\pi}x_{2j}, \quad \frac{dx_{2j}}{dw} = -\frac{2K(k_1)}{\pi}\mu_{02}x_{1j}; \quad j = 1, 2 \tag{4.6}$$

and the initial conditions

$$x_{11}(0) = x_{22}(0) = 1, \quad x_{12}(0) = x_{21}(0) = 0 \tag{4.7}$$

The quadratic part (4.4) of integral (4.3) is the first integral of linear equations (4.5). Using this integral, Eqs (4.6) can be integrated in explicit form and we obtain the following expressions for the quantities $x_{ij}(w)$

$$\begin{aligned} x_{11} &= \mathrm{cn}\upsilon\,\mathrm{dn}\upsilon\, f(\upsilon), \quad x_{21} = 2\mathrm{sn}\upsilon\,[4k_1^2\mathrm{sn}^2\upsilon - (1+k_1^2)(1+k_1^2\mathrm{sn}^4\upsilon)]f^3(\upsilon) \\ x_{12} &= \mathrm{sn}\upsilon\, f(\upsilon)/2, \quad x_{22} = \mathrm{cn}\upsilon\,\mathrm{dn}\upsilon\,(1+k_1^2\mathrm{sn}^4\upsilon)f^3(\upsilon) \\ f(\upsilon) &= (1-k_1^2\mathrm{sn}^4\upsilon)^{-1/2}, \quad \upsilon = K(k_1)w/\pi \end{aligned} \tag{4.8}$$

Here the modulus of the elliptic functions is equal to k_1.

When $w = \pi$ the matrix $\mathbf{X}(w)$ will have the following form

$$\mathbf{X}(\pi) = \begin{Vmatrix} 0 & a^{-2} \\ -a^2 & 0 \end{Vmatrix}, \quad a = [4(1-k_1^2)]^{1/4} \tag{4.9}$$

The roots (multipliers) of the characteristic equation of this matrix

$$\mu^2 + 1 = 0 \tag{4.10}$$

are different and have moduli equal to unity ($\mu_1 = i$ and $\mu_2 = -i$). Hence it follows [10] that the plane oscillations of the rigid body being investigated are orbitally stable in the linear approximation.

Calculation of the characteristic exponents. Resonance. Suppose $\pm i\lambda$ are the characteristic exponents of linear system (4.5). It follows from the equations $\mu_{1,2} = \exp(\pm i\pi\lambda)$ that λ will be the root of the equation $\cos\pi\lambda = 0$. Hence it follows that λ is a constant semi-integer number, which is independent of the amplitude of the oscillations of the body in unperturbed motion. The specific value of this number can be obtained using the continuous dependence of the characteristic exponents on the quantity k_1. To do this we consider oscillations of infinitesimal amplitude ($k_1 \to 0$). In the limiting case, when $k_1 = 0$, the function h_2 is the Hamiltonian $(p_2^2 + 4q_2^2)/8$ of a linear oscillator with frequency equal to $1/2$. Consequently, $\lambda = 1/2$. Hence, bearing in mind the π-periodicity of the Hamiltonian of the perturbed motion with respect to w, we obtain that the problem of the orbital stability of the plane oscillations of a rigid body in the Goryachev–Chaplygin case is always a resonance problem: for any amplitude of the oscillations fourth-order resonance ($4\lambda = 2$) occurs.

The orbital instability of plane oscillations in the strict non-linear formulation of the problem. According to the algorithm obtained earlier in [11], to investigate the non-linear problem of the orbital stability of the periodic motions of the body considered, it is necessary to obtain the normal form of the Hamiltonian of the perturbed motion (4.1).

Normalization of the quadratic part of the Hamiltonian (4.1). We must first construct a transformation which normalizes the Hamiltonian $h_2(q_2, p_2, w)$ of linear system (4.5). To do this (see [11]) we must replace the variables $w, r_1, q_2, p_2 \to u_1, \upsilon_1, u_2, \upsilon_2$ using the formulae

$$\begin{aligned}
w &= u_1, \quad r_1 = \upsilon_1 + \frac{1}{4}(u_2^2 + \upsilon_2^2) - h_2(n_{11}u_2 + n_{12}\upsilon_2, n_{21}u_2 + n_{22}\upsilon_2, u_1) \\
q_2 &= n_{11}u_2 + n_{12}\upsilon_2, \quad p_2 = n_{21}u_2 + n_{22}\upsilon_2 \\
n_{j1} &= \frac{1}{a}x_{j1}\cos\frac{u_1}{2} + ax_{j2}\sin\frac{u_1}{2}, \quad n_{j2} = -\frac{1}{a}x_{j1}\sin\frac{u_1}{2} + ax_{j2}\cos\frac{u_1}{2}, \quad j = 1, 2
\end{aligned} \tag{4.11}$$

where the functions $x_{ij}(u_1)$ are specified by formulae (4.8).

Transformation (4.11) is canonical, univalent and π-periodic in u_1. After the replacement (4.11), Hamilton's function (4.1) can be written in the form

$$\begin{aligned}
F &= F_2 + F_4 + \dots \\
F_2 &= \upsilon_1 + \frac{1}{4}(u_2^2 + \upsilon_2^2), \quad F_4 = \mu_{20}\upsilon_1^2 + f_2(u_2, \upsilon_2, u_1)\upsilon_1 + f_4(u_2, \upsilon_2, u_1)
\end{aligned} \tag{4.12}$$

Here f_k is a form of power k in u_2, υ_2 with π-periodic coefficients in u_1

$$f_k = \sum_{\nu+\mu=k} f_{\nu\mu}(u_1)u_2^\nu \upsilon_2^\mu$$

We will write the expressions required later for the coefficients of the form f_4

$$\begin{aligned}
f_{40} &= \mu_{20}m_{20}^2 + \mu_{12}m_{20}n_{11}^2 + \mu_{04}n_{11}^4 \\
f_{31} &= 2\mu_{20}m_{20}m_{11} + \mu_{12}(m_{11}n_{11}^2 + 2m_{20}n_{11}n_{12}) + 4\mu_{04}n_{11}^3 n_{12} \\
f_{22} &= \mu_{20}(m_{11}^2 + 2m_{20}m_{02}) + \mu_{12}(m_{02}n_{11}^2 + 2m_{11}n_{11}n_{12} + m_{20}n_{12}^2) + 6\mu_{04}n_{11}^2 n_{12}^2 \\
f_{13} &= 2\mu_{20}m_{11}m_{02} + \mu_{12}(m_{11}n_{12}^2 + 2m_{02}n_{11}n_{12}) + 4\mu_{04}n_{11}n_{12}^3 \\
f_{04} &= \mu_{20}m_{02}^2 + \mu_{12}m_{02}n_{12}^2 + \mu_{04}n_{12}^4 \\
m_{20} &= 1/4 - K(k_1)(4\mu_{02}n_{11}^2 + n_{21}^2)/(4\pi), \quad m_{02} = 1/4 - K(k_1)(4\mu_{02}n_{12}^2 + n_{22}^2)/(4\pi) \\
m_{11} &= -K(k_1)(4\mu_{02}n_{11}n_{12} + n_{21}n_{22})/(2\pi)
\end{aligned} \tag{4.13}$$

In the new variables, integral (4.3) can be written in the form of a series in powers of the quantities u_2, υ_2 and υ_1

$$G = G_2 + G_4 + \dots = c \tag{4.14}$$

where G_n is a form of power n in u_2, υ_2 and $|\upsilon_1|^{1/2}$, where calculations show that

$$G_2 = 4k_1\sqrt{1-k_1^2}\left[\left(\sin\frac{u_1}{2}u_2 + \cos\frac{u_1}{2}\upsilon_2\right)^2 - \left(\cos\frac{u_1}{2}u_2 - \sin\frac{u_1}{2}\upsilon_2\right)^2\right] \quad (4.15)$$

The dots in relations (4.12) and (4.14) denote a set of terms of higher than the fifth power in u_2, υ_2 and $|\upsilon_1|^{1/2}$.

Normalization of the Hamiltonian of the perturbed motion up to terms of the fourth power inclusive. We can obtain the normal form of terms of the fourth power in Hamilton's function (4.12), following the approach described earlier in [11], using the Deprit–Hori method [12, 13]. The normalizing canonical transformation $u_1, \upsilon_1, u_2, \upsilon_2 \to \theta_1, \rho_1, \xi_2, \eta_2$ will be close to identical to

$$u_1 = \theta_1 + \dots, \quad \upsilon_1 = \rho_1 + \dots, \quad u_2 = \xi_2 + \dots, \quad \upsilon_2 = \eta_2 + \dots$$

The dots denote convergent series in powers of ρ_1, ξ_2, η_2 with coefficients that are π-periodic in θ_1.

The normalized Hamiltonian (4.12) takes the form

$$\Gamma = \rho_1 + \frac{1}{2}\rho_2 + c_{20}\rho_1^2 + c_{11}\rho_1\rho_2 + \rho_2^2[c_{02} + \alpha_{40}\sin(4\theta_2 - 2\theta_1) + \beta_{40}\cos(4\theta_2 - 2\theta_1)] + O_3; \quad (4.16)$$

$$\xi_2 = \sqrt{2\rho_2}\sin\theta_2, \quad \eta_2 = \sqrt{2\rho_2}\cos\theta_2$$

The constant coefficients c_{ij}, α_{40} and β_{40} are calculated from the formulae [11]

$$c_{20} = \mu_{20}, \quad c_{11} = \langle f_{20} + f_{02}\rangle, \quad c_{02} = \frac{1}{2}\langle 3f_{40} + f_{22} + 3f_{04}\rangle$$

$$\alpha_{40} = -\frac{1}{2}\langle \sigma_{40}\sin 2u_1 - \chi_{40}\cos 2u_1\rangle, \quad \beta_{40} = \frac{1}{2}\langle \sigma_{40}\cos 2u_1 + \chi_{40}\sin 2u_1\rangle \quad (4.17)$$

$$\sigma_{40} = f_{40} - f_{22} + f_{04}, \quad \chi_{40} = f_{13} - f_{31}$$

The symbol $\langle g\rangle$ denotes the mean value of the π-periodic function $g(u_1)$ over a period.

Apart from the unimportant constant factor $8k_1\sqrt{1-k_1^2}$, integral (4.14) can be written in the form

$$G = \rho_2\cos(2\theta_2 - \theta_1) + O_2 = \text{const} \quad (4.18)$$

We have denoted by O_k the set of terms not lower than the k-th power in ρ_1 and ρ_2.

It can be seen from relations (4.2), (4.8), (4.13) and (4.17) that σ_{40} is an even function of u_1 while χ_{40} is an odd function of u_1. Hence, the coefficient α_{40} of normal form (4.16) is equal to zero.

We can simplify the structure of normal form (4.16) and integral (4.18) somewhat by making the equivalent canonical replacement of variables in accordance with the formulae

$$\theta_1 = \psi_1, \quad \theta_2 = \psi_1/2 + \psi_2, \quad \rho_1 = R_1 - R_2/2, \quad \rho_2 = R_2 \quad (4.19)$$

Taking into account the fact that $\alpha_{40} = 0$, we have in the new variables

$$\Gamma = R_1 + a_{20}R_1^2 + a_{11}R_1R_2 + (a_{02} + \beta_{40}\cos 4\psi_2)R_2^2 + O_3 \quad (4.20)$$

$$a_{20} = c_{20}, \quad a_{11} = c_{11} - c_{20}, \quad a_{02} = c_{02} - c_{11}/2 + c_{20}/4$$
$$G = R_2\cos 2\psi_2 + O_2 = \text{const} \quad (4.21)$$

The properties of the coefficients of normal form (4.20), which arise from the existence of integral (4.21). The sufficient conditions for stability and instability of a system with Hamilton function of the form (4.21) are well known [13]: if the inequality $|a_{02}| > |\beta_{40}|$ is satisfied, the system is stable, and if $|a_{02}| < |\beta_{40}|$, the system is unstable. The coefficients of the normal form are calculated from (4.17). But in the specific problem of the stability of the plane oscillations of a rigid body being investigated here, the equations of perturbed motion have the integral (4.21). Hence, as previously, without having to carry out calculations using these formulae, it can be shown that the coefficients of normal form (4.20) are not completely arbitrary, but must satisfy certain relations.

In fact, the condition that (4.21) is an integral can be written in the form of the Poisson bracket of the functions G and Γ being equal to zero

$$\sum_{i=1}^{2}\left(\frac{\partial G}{\partial \psi_i}\frac{\partial \Gamma}{\partial R_i}-\frac{\partial G}{\partial R_i}\frac{\partial \Gamma}{\partial \psi_i}\right) = 0$$

It follows from relations (4.20) and (4.21) that this condition can be written in the form of an equality

$$\sin 2\psi_2 R_2[a_{11}R_1 + 2(a_{02}-\beta_{40})R_2] + O_3 = 0$$

which must be satisfied for any values of ψ_1, ψ_2 and arbitrary fairly small R_1 and R_2. This is only possible when the following relations are satisfied

$$a_{11} = 0, \quad a_{02} = \beta_{40}$$

Consequently, the normalized Hamiltonian (4.20) has the following structure

$$\Gamma = R_1 + a_{20}R_1^2 + \beta_{40}(1+\cos 4\psi_2)R_2^2 + O_3 \tag{4.22}$$

Computer calculations using formulae (4.17) showed that the coefficient β_{40} of normal form (4.22) is a positive function of k_1 (i.e. of the amplitude of the oscillations of a rigid body in unperturbed motion). As $k_1 \to 0$ the function β_{40} approaches zero, and as $k_1 \to 1$ it increases without limit. A graph of the function $\beta_{40} = \beta_{40}(k_1)$ is shown in Fig. 1.

Proof of the orbital instability of plane oscillations. For Hamilton's function (4.22) the sufficient conditions for stability and instability formulated above are not satisfied. And, consequently, for any amplitude of the oscillations of the body in unperturbed motion the critical case of fourth-order resonance is obtained [14, 15]. In this case the approximate system, Hamilton's function of which is obtained from the function (4.22) by dropping the terms O_3, is unstable. Generally speaking, it should be possible to choose the terms O_3 so that the total system remains unstable or, conversely, converts into a stable state [14, 15]. But, as can be seen from what follows, the latter is impossible in the specific problem considered, and this is due to the existence in the equations of perturbed motion of an integral of the form (4.21).

To prove the orbital instability of the motion of the rigid body investigated, it is sufficient to show its instability at the zeroth isoenergy level $\Gamma = 0$.

We will take the coordinate ψ_1 as the independent variable. It follows from the equations of motion corresponding to Hamilton's function (4.22) that, in a fairly small neighbourhood of the unperturbed trajectory, the variable ψ_1 will be a monotonically increasing function of time and, consequently, it can play the same role as the time in the instability problem.

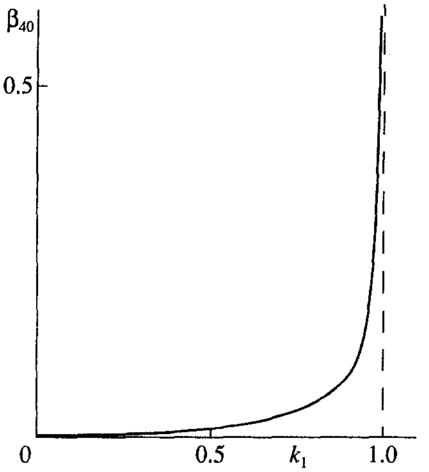

Fig. 1

From the equation $\Gamma = 0$ for small R_1 and R_2 we obtain

$$R_1 = -S = -\beta_{40}(1+\cos 4\psi_2)R_2^2 + \tilde{S}(R_2, \psi_2, \psi_1), \quad \tilde{S} = O(R_2^3) \tag{4.23}$$

At the isoenergy level $\Gamma = 0$ the perturbed motion described by Whittaker's equations [16], which have the Hamilton form

$$\begin{aligned} \frac{d\psi_2}{d\psi_1} &= \frac{\partial S}{\partial R_2} = 2\beta_{40}(1+\cos 4\psi_2)R_2 + O(R_2^2), \\ \frac{dR_2}{d\psi_1} &= -\frac{\partial S}{\partial \psi_2} = 4\beta_{40}\sin 4\psi_2 R_2^2 + O(R_2^3) \end{aligned} \tag{4.24}$$

These equations have the integral

$$G(R_2, \psi_2, \psi_1) = R_2\cos 2\psi_2 + \tilde{G}(R_2, \psi_2, \psi_1) = \text{const}, \quad \tilde{G} = O(R_2^2) \tag{4.25}$$

which is obtained from integral (4.21) if we replace the quantity R_1 in it by its value defined by relations (4.23)

To prove the instability we will use Chetayev's theorem [10]. We will take Chetayev's function in the form

$$V = R_2^2 \sin 4\psi_2 \tag{4.26}$$

We will take the region $0 < \psi_2 < \pi/4$ as the region $V > 0$.

By virtue of the equations of motion (4.24) we obtain the following expression for the derivative of the function (4.26)

$$\frac{dV}{d\psi_1} = 8\beta_{40}(1+\cos 4\psi_2)R_2^3 + O(R_2^4) \tag{4.27}$$

For the trajectories $\psi_2 = \psi_2(\psi_1)$, $R_2 = R_2(\psi_1)$ of system (4.24), which begin in the region $V > 0$ for fairly small values of $R_2(0)$, the quantity $G_0 = G(R_2(0), \psi_2(0), 0)$ of constant integral (4.25) is non-zero. It follows from the relation $G = G_0$ that $R_2\cos 2\psi_2 = G_0 + O(R_2^2)$. Taking this equation into account, expression (4.27) for the derivative $dV/d\psi_1$ can be represented in the form

$$\frac{dV}{d\psi_1} = 16\beta_{40}R_2[G_0^2 + G_0 O(R_2^2) + O(R_2^3)]$$

For sufficiently small R_2 the derivative $dV/d\psi_1$ is positive. Consequently, by Chetayev's theorem, we have instability.

5. PENDULUM-LIKE ROTATIONS OF THE BODY

In the case of rotations, Hamilton's function of perturbed motion, as in the case of oscillations, can be represented by a series of the form (4.1), where now the quantity $\omega_2\tau$ is taken as the independent variable, while the function $h_2(q_2, p_2, w)$ is given by the equalities

$$h_2 = \frac{k_2 K(k_2)}{8\pi}[p_2^2 + 4\nu_{02}q_2^2], \quad \nu_{02} = k_2^{-2}(3\,\mathrm{dn}^2 u + k_2^2 - 2) \tag{5.1}$$

The quantity u is defined by the last of relations (3.4) and k_2 corresponds to the unperturbed motion. The terms of the expansion (4.1) of powers higher than the third in q_2, p_2 and $|r_1|^{1/2}$, are not written, since their explicit expression is not required in what follows. The function (4.1) is 2π-periodic in w.

We will write integral (4.2) in the form of series (4.3), where

$$g_2 = 8k_2^{-3}\mathrm{dn}u(1 - k_2^2\mathrm{cn}^2 u)q_2^2 + 8\,\mathrm{sn}u\,\mathrm{cn}u\,q_2 p_2 + 2k_2^{-1}\mathrm{dn}u\,p_2^2 \tag{5.2}$$

We can obtain the following explicit expressions for the elements $x_{ij}(w)$ of the matrix of fundamental solutions $\mathbf{X}(w)$ of system (4.5)

$$
\begin{aligned}
x_{11} &= \mathrm{cn}\nu\mathrm{dn}\nu g(\nu), \quad x_{21} = 2k_2^{-1}\mathrm{sn}\nu[4k_2^2\mathrm{sn}^2\nu - (1+k_2^2)(1+k_2^2\mathrm{sn}^4\nu)]g^3(\nu) \\
x_{12} &= k_2/2\ \mathrm{sn}\nu g(\nu), \quad x_{22} = \mathrm{cn}\nu\mathrm{dn}\nu(1+k_2^2\mathrm{sn}^4\nu)g^3(\nu) \qquad (5.3) \\
g(\nu) &= (1-k_2^2\mathrm{sn}^4\nu)^{-1/2}, \quad \nu = K(k_2)w/(2\pi)
\end{aligned}
$$

In expressions (5.3) the modulus of the elliptic functions is equal to k_2.

The matrix $\mathbf{X}(2\pi)$ will be

$$
\mathbf{X}(2\pi) = \left\| \begin{matrix} 0 & b^{-2} \\ -b^2 & 0 \end{matrix} \right\|, \quad b = \left[\frac{4(1-k_2^2)}{k_2^2}\right]^{1/4} \qquad (5.4)
$$

Its characteristic equation has the form (4.10). Hence, as in the case of oscillations, we have $\mu_1 = i$ and $\mu_2 = -i$, and, consequently, pendulum-like rotations of the body are orbitally stable in the linear approximation.

The imaginary part of the characteristic exponents $\pm i\lambda$ satisfies the equation $\cos 2\pi\lambda = 0$. Hence, the quantity λ is uniquely defined. We can only assert that it is independent of the angular velocity of rotation of the body in unperturbed motion and differs from an integer by an amount equal to 1/4. The non-uniqueness is eliminated by considering infinitely large angular velocities ($k_2 \to 0$). If we make the canonical replacement of variables in Hamiltonian (5.1) then, in the limit $k_2 = 0$, we obtain the Hamiltonian $(\tilde{p}_2^2 + \tilde{q}_2^2)/8$ of a linear oscillator with frequency equal to 1/4. Consequently, $\lambda = 1/4$. Hence, in the case of rotations, as in the case of oscillations considered above, we always have fourth-order resonance.

However, unlike the case of oscillations, the pendulum-like rotations of a rigid body are orbitally stable in the Goryachev–Chaplygin case. In order to prove this we will use Rumyantsev's theorem on the stability with respect to some of the variables [17]. We obtain the function V as a quadratic combination of the integral H = const and the additional integral (4.3).

It is convenient first of all to make the canonical replacement of variables $w, r_1, q_2, p_2 \to u_1, \upsilon_1, u_2, \upsilon_2$, which is 2π-periodic in u_1, which normalizes the Hamiltonian of the linear system of equations of the perturbed motion. This replacement is given by equalities similar to (4.11). We only need to replace the coefficient 1/4 in the equality for r_1 by the coefficient 1/8, and the coefficients n_{ij} must now be calculated from the following formulae

$$
n_{j1} = \frac{1}{b}x_{j1}\cos\frac{u_1}{4} + bx_{j2}\sin\frac{u_1}{4}, \quad n_{j2} = -\frac{1}{b}x_{j1}\sin\frac{u_1}{4} + bx_{j2}\cos\frac{u_1}{4}, \quad j = 1, 2
$$

The functions $x_{ij}(u_1)$ are given by Eqs (5.3).

In the new variables, the Hamiltonian of the perturbed motion takes the following form

$$
F = \upsilon_1 + \frac{1}{8}(u_2^2 + \upsilon_2^2) + \ldots \qquad (5.5)
$$

while integral (4.3), apart from the constant factor $4\sqrt{1-k_2^2}/k_2^2$, can be written in the form

$$
G = u_2^2 + \upsilon_2^2 + \ldots = \mathrm{const} \qquad (5.6)
$$

In Eqs (5.5) and (5.6) the dots denote terms of series the power of which in u_2, υ_2, $|\upsilon_1|^{1/2}$ is higher than the third.

Suppose

$$
V = F^2 + G^2 \qquad (5.7)
$$

The function $V \geqslant 0$, where $V = 0$ only for those values of the variables v_1, u_2 and v_2 which satisfy the system of two equations

$$F = 0, \quad G = 0 \tag{5.8}$$

From the first equation we can express the quantity v_1 in terms of u_2, v_2 and u_1. We obtain

$$v_1 = -\frac{1}{8}(u_2^2 + v_2^2) + \tilde{R}_1(u_2, v_2, u_1), \quad \tilde{R}_1 = O((u_2^2 + v_2^2)^2) \tag{5.9}$$

Taking this equation into account, the second equation of system (5.8) can be written in the form

$$u_2^2 + v_2^2 + \tilde{G}(u_2, v_2, u_1) = 0, \quad \tilde{G} = O((u_2^2 + v_2^2)^2)$$

For sufficiently small u_2 and v_2 this equation is satisfied only when $u_2 = v_2 = 0$. It follows from (5.9) that $v_1 = 0$. Hence, the function V is positive-definite in the variables v_1, u_2 and v_2 and, by Rumyantsev's theorem, the unperturbed motion is stable with respect to these variables. This also indicates that the pendulum-like rotations of a rigid body in the Goryachev–Chaplygin case are orbitally stable.

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